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Convex polyominoes and algebraic languages

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Abstract. We use the so-called DSV methodology, that links some enumeration problems to the theory of algebraic languages, to get a system of q-difference equations involving the generating functions of convex polyominoes, of convex and directed polyominoes, and of parallelogram polyominoes, according to their height, width and area. Then, we show various applications of this system.

1. Introduction

Polyominoes are classical objects in combinatorics. They are also studied in physics as special cases of self-avoiding polygons, that are used to model crystal growth or polymers. A polyomino is a finite union of elementary cells having its interior connected (figure 1).

We consider here several subclasses of convex polyominoes (figure 2), namely the parallelogram polyominoes, the directed and convex polyominoes, and finally the stack polyominoes, that have been defined in the introduction of the previous paper (Bousquet-Mélou 1992), where we also recalled some results concerning the enumeration of these objects.

We use the so-called DSV methodology, which relates some enumerative problems to the theory of algebraic languages to get a system of q-difference equations involving the generating functions of parallelogram polyominoes, directed and convex polyominoes, and convex polyominoes, according to their height, width and area (Bousquet-Mélou 1990).

2. Algebraic languages and DSV methodology

The concept of algebraic language is very classical in theoretical computer science.

Let X be an alphabet, that is a finite and non-empty set. The elements of X are called letters. A word on X is a finite sequence $a_1 \dots a_n$, where $a_1 \dots a_n$ are letters of X. The empty word is denoted e. The set of words on X is denoted X^*. We define a product (or concatenation) on X^*: if u is the word $a_1 \dots a_n$ and v the word $b_1 \dots b_m$, then the product of u and v is $a_1 \dots a_n b_1 \dots b_m$. This operation is not commutative.

If a is an element of X, we denote $|u|_a$ the number of letters a in the word u.

A language $\mathscr{L}$ is a subset of X^*. Intuitively, it is said to be algebraic when a set of rewriting-rules, applied recursively, allow to form all the words of $\mathscr{L}$, and no other. This set of rules is a grammar, and is non-ambiguous when any word of $\mathscr{L}$ can be obtained in a unique way, using the rewriting-rules.

The most famous—and simple—algebraic language is the Dyck language, denoted $\mathscr{D}$. Let X be the alphabet $\{x, \bar{x}\}$. Then $\mathscr{D}$ is the set of words u on X satisfying the two

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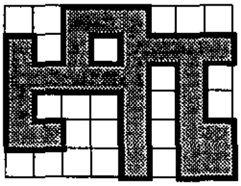

Figure 1. A polyomino.

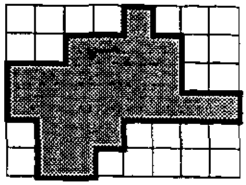

Figure 2. A convex polyomino.

following conditions: (i) $|u|_x = |u|_{\bar{x}}$; (ii) if u is factorized as vw, where v and w are words of X^*, then $|v|_x \geqslant |v|_{\bar{x}}$.

This language is generated by the grammar composed of the two rewriting rules:

(a) $D \to e$ (empty word)

(b) $D \to xD\bar{x}D$.

For example, the word $xx\bar{x}x\bar{x}\bar{x}x\bar{x}$ is produced by the sequence bbabaabaa. At each state, we apply the chosen rewriting-rule on the leftmost letter 'D' in the word.

(b) $D \to xD\bar{x}D$

(b) $\to xxD\bar{x}D\bar{x}D$

(a) $\to xx\bar{x}D\bar{x}D$

(b) $\to xx\bar{x}xD\bar{x}D\bar{x}D$

(a) $\to xx\bar{x}x\bar{x}D\bar{x}D$

(a) $\to xx\bar{x}x\bar{x}\bar{x}D$

(b) $\to xx\bar{x}x\bar{x}\bar{x}xD\bar{x}D$

(a) $\to xx\bar{x}x\bar{x}\bar{x}x\bar{x}D$

(a) $\to xx\bar{x}x\bar{x}\bar{x}x\bar{x}$.

This grammar is non-ambiguous since any non-empty Dyck word has a unique factorization $xv\bar{x}w$, where v and w are Dyck words.

For any language $\mathscr{L}$, we associate its formal generating function $\underline{\underline{\mathscr{L}}} = \Sigma_{u \in \mathscr{L}}\, u$.

If $\mathscr{L}$ is algebraic and generated by a non-ambiguous grammar, then $\underline{\underline{\mathscr{L}}}$ is a component of the unique solution of an algebraic (non-commutative) system. For example, we deduce from the rewriting-rules generating the Dyck language that the generating function $\underline{\underline{\mathscr{D}}}$ of Dyck words satisfies the following algebraic equation:

$$\underline{\underline{\mathscr{D}}} = e + x\underline{\underline{\mathscr{D}}}\bar{x}\underline{\underline{\mathscr{D}}}.$$

About fifteen years ago Schützenberger introduced a new idea that linked algebraic languages and some enumeration problems. His idea was at first used to explain the algebraic character of some (already known) generating functions (see Cori 1970, 1972, 1975), and then to obtain new results. The first one was the following: the number of convex polyominoes having perimeter $2n+8$ is $(2n+11)4^n - 4(2n+1)\binom{2n}{n}$ (Delest and Viennot 1984).

The DSV method consists in building a bijection between the objects one wishes to enumerate and words of an algebraic language $\mathscr{L}$, so that the size of an object is the length (the number of letters) of the associated word. If $\mathscr{L}$ is generated by a non-ambiguous grammar, we can derive from the algebraic system on $\underline{\underline{\mathscr{L}}}$ another algebraic system, which is commutative and involves the generating function of the objects. More details and examples can be found in the works of Delest and Viennot (1984) and Viennot (1985).

Several refinements of this methodology were found to get multivariate (but still algebraic) generating functions. Recently, Delest and Fedou (1989) introduced a 'q-analogue' of this methodology, that allows them to obtain even non-algebraic series. They used it successfully to enumerate parallelogram polyominoes according to their width and area (Fedou 1989).

We generalize their work by using another coding that Delest (1988) constructed at first to enumerate column-convex polyominoes. We finally get a system of q-difference equations involving the generating functions of parallelogram polyominoes, directed and convex polyominoes, and convex polyominoes.

Notation. The generating function of a given subset $\mathscr{S}$ of polyominoes will be

$$P(x, y, q) = \sum_{n,m,a} x^n y^m q^a P_{n,m,a}$$

where $P_{n,m,a}$ is the number of polyominoes of $\mathscr{S}$ having width n, height m and area a.

3. q-difference equations

A polyomino is column-convex if its intersection with any vertical line is connected. We describe at first Delest's coding for column-convex polyominoes.

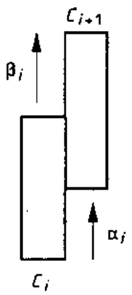

Figure 3. The α_i and β_i numbers.

Let P be a column-convex polyomino having m columns, denoted, from left to right, $C_1, \ldots, C_m$. For $1 \leqslant i \leqslant m-1$, let α_i (respectively β_i) be the difference between the bottom (the top) of C_{i+1} and the bottom (top) of C_i. Let $\alpha_0 = 0$, $\beta_m = 0$, $\alpha_m = h_m - 1$ and $\beta_0 = h_1 - 1$, where h_1 (h_m) is the height of C_1 (C_m). Note that α_i and β_i can either be positive or not (figure 3).

Let $g(P)$ be the word $u_0 x\bar{x} u_1 x\bar{x} \ldots x\bar{x} u_m$, with, for $0 \leqslant i \leqslant m$:

$$u_i = \begin{cases} \bar{x}^{\alpha_i} x^{\beta_i} & \text{if } \alpha_i \geqslant 0 \text{ and } \beta_i \geqslant 0 \\ \bar{x}^{\alpha_i} \bar{y}^{|\beta_i|} & \text{if } \alpha_i \geqslant 0 \text{ and } \beta_i \leqslant 0 \\ y^{|\alpha_i|} x^{\beta_i} & \text{if } \alpha_i \leqslant 0 \text{ and } \beta_i \geqslant 0 \\ \bar{y}^{|\beta_i|} y^{|\alpha_i|} & \text{if } \alpha_i \leqslant 0 \text{ and } \beta_i \leqslant 0. \end{cases}$$

Let $\mathscr{V}$ be the set of all words $g(P)$, when P describes the set of column-convex polyominoes. Then g is a bijection between column-convex polyominoes and words of $\mathscr{V}$ (Delest 1988).

Let u be a word of $\mathscr{V}$. If $u = va\bar{b}w$, where a and b are elements of $\{x, y\}$, the factor $a\bar{b}$ is call a peak of u, and the height of this peak is $1+|v|_x+|v|_y-|v|_{\bar{x}}-|v|_{\bar{y}}$. The perimeter, width and area of a column-convex polyomino P can be read on $g(P)$ as follows:

- the perimeter of P is the length (number of letters) of $g(P)$, plus two,
- the width of P is the number of peaks of $g(P)$,
- the area of P is the sum of the heights of the peaks of $g(P)$.

Convex polyominoes are obviously special cases of column-convex polyominoes. So we can apply our coding g to each studied subset of convex polyominoes, and try to find out whether its image is an algebraic language generated by a non-ambiguous grammar. In this case, we will get a system of equations involving the generating series of this subset.

We first notice that the image under g of the set of parallelogram polyominoes is the set of non-empty Dyck words. The image under g of the set of directed and convex polyominoes is also an algebraic language generated by a non-ambiguous grammar. The case of convex polyominoes is more difficult: actually, we do not know whether its image under g is an algebraic language or not. Nevertheless, we can define two subsets of convex polyominoes, denoted $\mathscr{A}$ and $\mathscr{B}$, so that:

- to enumerate $\mathscr{A}$ and $\mathscr{B}$ is enough to enumerate convex polyominoes,
- the images under g of $\mathscr{A}$ and $\mathscr{B}$ are both generated by an algebraic non-ambiguous grammar.

Let P be a convex polyomino and R be the smallest rectangle containing P. Let [N, N′] (respectively [W, W′], [S, S′], [E, E′]) be the intersection of P with the upper (respectively left, lower, right) border of R, the points N, N′, W, W′, S, S′, E, E′ being taken counterclockwise (figure 4).

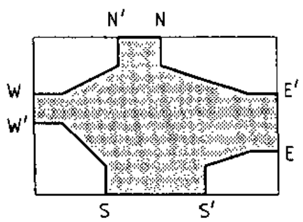

Figure 4. A convex polyomino.

As in the previous paper (Bousquet-Mélou 1992), we define three subsets of convex polyominoes. Let $\mathscr{A}$ be the set of convex polyominoes such that the vertical line passing by N is at the right of the vertical line by S. Let $\mathscr{A}'$ be the set of convex polyominoes such that the vertical line passing by S' is at the right of the vertical line passing by N' (figure 5). Let $\mathscr{B}$ be the intersection of $\mathscr{A}$ and $\mathscr{A}'$.

Note that the symmetric, up to any vertical axis, of a polyomino of $\mathscr{A}$ is a polyomino of $\mathscr{A}'$ (and vice-versa), and that the union of $\mathscr{A}$ and $\mathscr{A}'$ is the set of convex polyominoes.

These remarks imply that the generating function $Z(x, y, q)$ of convex polyominoes is

$$Z = 2A - B \tag{1}$$

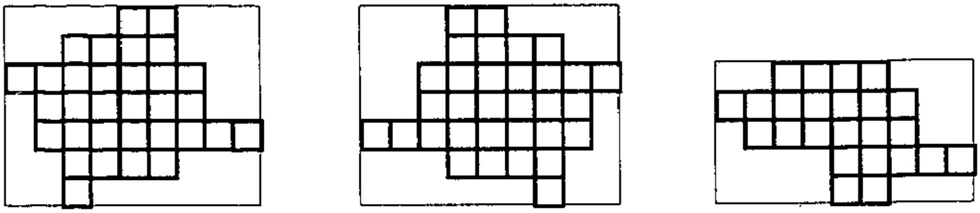

Figure 5. Elements of (left to right) $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{B}$.

where $A(x, y, q)$ (respectively $B(x, y, q)$) is the generating function of the convex polyominoes of $\mathcal{A}$ ($\mathcal{B}$).

Let $\mathcal{D}$ (respectively $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$) be the image under g of the set of parallelogram polyominoes (respectively directed and convex polyominoes, polyominoes of $\mathcal{A}$, polyominoes of $\mathcal{B}$). Note that we changed the previous notations, since $\mathcal{D}$ is now the set of non-empty Dyck words.

The images under g of the sets $\mathcal{A}$ and $\mathcal{B}$ are algebraic languages generated by non-ambiguous grammars. We introduce nine other languages, namely $\mathcal{E}_1$, $\mathcal{E}'$, $\mathcal{E}'_2$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$, to determine the rewriting-rules of the grammars generating $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$. We deduce from these rewriting-rules that the following system has a unique solution, denoted ($\underline{\underline{\mathcal{D}}}$, $\underline{\underline{\mathcal{E}}}$, $\underline{\underline{\mathcal{E}_1}}$, $\underline{\underline{\mathcal{E}'}}$, $\underline{\underline{\mathcal{E}'_2}}$, $\underline{\underline{\mathcal{F}}}$, $\underline{\underline{\mathcal{F}_1}}$, $\underline{\underline{\mathcal{F}_2}}$, $\underline{\underline{\mathcal{F}_3}}$, $\underline{\underline{\mathcal{G}}}$, $\underline{\underline{\mathcal{G}_1}}$, $\underline{\underline{\mathcal{G}_2}}$, $\underline{\underline{\mathcal{G}_3}}$) and that $\underline{\underline{\mathcal{D}}}$ (respectively $\underline{\underline{\mathcal{E}}}$, $\underline{\underline{\mathcal{F}}}$, $\underline{\underline{\mathcal{G}}}$) is the formal generating function of $\mathcal{D}$ (respectively $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$):

$$\begin{aligned}
&\underline{\underline{\mathcal{D}}} = x\bar{x} + x\bar{x}\underline{\underline{\mathcal{D}}} + x\underline{\underline{\mathcal{D}}}\bar{x} + x\underline{\underline{\mathcal{D}}}\bar{x}\underline{\underline{\mathcal{D}}}\\
&\underline{\underline{\mathcal{E}}} = x\bar{x} + x\bar{x}\underline{\underline{\mathcal{E}_1}} + x\underline{\underline{\mathcal{E}}}\bar{x} + x\underline{\underline{\mathcal{E}}}\bar{x}\underline{\underline{\mathcal{D}}} \qquad \underline{\underline{\mathcal{E}_1}} = \underline{\underline{\mathcal{E}}} + y\underline{\underline{\mathcal{E}_1}}\bar{x} + y\underline{\underline{\mathcal{E}_1}}\bar{x}\underline{\underline{\mathcal{D}}}\\
&\underline{\underline{\mathcal{E}'}} = x\bar{x} + \underline{\underline{\mathcal{E}'_2}}x\bar{x} + x\underline{\underline{\mathcal{E}'}}\bar{x} + \underline{\underline{\mathcal{D}}}x\underline{\underline{\mathcal{E}'}}\bar{x} \qquad \underline{\underline{\mathcal{E}'_2}} = \underline{\underline{\mathcal{E}'}} + x\underline{\underline{\mathcal{E}'_2}}\bar{y} + \underline{\underline{\mathcal{D}}}x\underline{\underline{\mathcal{E}'_2}}\bar{y}\\
&\underline{\underline{\mathcal{F}}} = x\bar{x} + x\bar{x}\underline{\underline{\mathcal{F}_1}} + x\underline{\underline{\mathcal{E}}}\bar{x}(e + \underline{\underline{\mathcal{E}'}}) + x(\underline{\underline{\mathcal{F}}} - \underline{\underline{\mathcal{E}}})\bar{x}\{x\bar{x}\}^* + x\underline{\underline{\mathcal{F}_2}}\bar{y}\{x\bar{x}\}^+\\
&\underline{\underline{\mathcal{F}_1}} = \underline{\underline{\mathcal{F}}} + y\underline{\underline{\mathcal{E}_1}}\bar{x}(e + \underline{\underline{\mathcal{E}'}}) + y(\underline{\underline{\mathcal{F}_1}} - \underline{\underline{\mathcal{E}_1}})\bar{x}\{x\bar{x}\}^* + y\underline{\underline{\mathcal{F}_3}}\bar{y}\{x\bar{x}\}^+ \qquad (2)\\
&\underline{\underline{\mathcal{F}_2}} = \underline{\underline{\mathcal{F}}} + (e + \underline{\underline{\mathcal{E}}})x\underline{\underline{\mathcal{E}'_2}}\bar{y} + \{x\bar{x}\}^*x(\underline{\underline{\mathcal{F}_2}} - \underline{\underline{\mathcal{E}'_2}})\bar{y} + \{x\bar{x}\}^+y\underline{\underline{\mathcal{F}_3}}\bar{y}\\
&\underline{\underline{\mathcal{F}_3}} = \underline{\underline{\mathcal{F}_2}} + y\underline{\underline{\mathcal{E}_1}}\bar{x}(e + \underline{\underline{\mathcal{E}'_2}}) + y(\underline{\underline{\mathcal{F}_1}} - \underline{\underline{\mathcal{E}_1}})\bar{x}\{x\bar{x}\}^* + y\underline{\underline{\mathcal{F}_3}}\bar{y}\{x\bar{x}\}^*\\
&\underline{\underline{\mathcal{G}}} = x\bar{x} + x\bar{x}\underline{\underline{\mathcal{G}_1}} + x\underline{\underline{\mathcal{G}}}\bar{x}\{x\bar{x}\}^* + x\underline{\underline{\mathcal{G}_2}}\bar{y}\{x\bar{x}\}^+ \qquad \underline{\underline{\mathcal{G}_1}} = \underline{\underline{\mathcal{G}}} + y\underline{\underline{\mathcal{G}_1}}\bar{x}\{x\bar{x}\}^* + y\underline{\underline{\mathcal{G}_3}}\bar{y}\{x\bar{x}\}^+\\
&\underline{\underline{\mathcal{G}_2}} = \underline{\underline{\mathcal{G}}} + \{x\bar{x}\}^*x\underline{\underline{\mathcal{G}_2}}\bar{y} + \{x\bar{x}\}^+y\underline{\underline{\mathcal{G}_3}}\bar{y} \qquad \underline{\underline{\mathcal{G}_3}} = \underline{\underline{\mathcal{G}_2}} + y\underline{\underline{\mathcal{G}_1}}\bar{x}\{x\bar{x}\}^* + y\underline{\underline{\mathcal{G}_3}}\bar{y}\{\bar{x}\}^*.
\end{aligned}$$

(The notation $\{x\bar{x}\}^+$ (respectively $\{x\bar{x}\}^*$) stands for $\Sigma_{n\geq 1}(x\bar{x})^n$ (respectively $\Sigma_{n\geq 0}(x\bar{x})^n$)).

We can now translate these equations into a commutative system involving the generating functions of the words of these languages according to their length, number of peaks, and sum of the heights of their peaks. For example, let

$$D(t^2, x, q) = \sum_{n,p,s} t^{2n}x^p q^s D_{n,p,s}$$

where $D_{n,p,s}$ is the number of Dyck words of length $2n$, having p peaks, such that the sum of the heights of these peaks is s. Then the first equation of system (2) leads to:

$$D(t^2, x, q) = t^2xq + t^2xqD(t^2, x, q) + t^2D(t^2, xq, q) + t^2D(t^2, xq, q)D(t^2, x, q). \qquad (3)$$

Then, thanks to the properties of the bijection g, the generating function of parallelogram polyominoes is $X(x, y, q) = yD(y, x/y, q)$. We thus have

$$X(x, y, q) = xyq + xqX(x, y, q) + (y + X(x, y, q))X(xq, y, q)$$

which will be written

$$X(x)=\frac{xyq}{1-xq}+\frac{y+X}{1-xq}X(xq). \qquad (4)$$

Using similar transformations on every equation of system (2), and also relation (1), we finally get the following matrical system. It has a unique solution where all the components are formal series in the three variables x, y and q, denoted (X, Y, Y_1, Z, Z_1, Z_3). Then X (respectively Y, Z) is the generating function of parallelogram (respectively directed and convex, convex) polyominoes:

$$X(x)=\frac{xyq}{1-xq}+\frac{y+X}{1-xq}X(xq)$$

$$\begin{pmatrix} Y \\ Y_1 \end{pmatrix}(x)=\frac{xyq}{1-xq}\begin{pmatrix} 1 \\ 1 \end{pmatrix}+\frac{y+X}{1-xq}\begin{pmatrix} 1 & xq \\ 1 & 1 \end{pmatrix}\begin{pmatrix} Y \\ Y_1 \end{pmatrix}(xq) \qquad (5)$$

$$\begin{pmatrix} Z \\ Z_1 \\ Z_3 \end{pmatrix}(x)=\frac{xyq}{1-xq}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}+\frac{2}{y+X}\begin{pmatrix} V^2 \\ VW \\ W^2 \end{pmatrix}+\frac{y}{(1-xq)^2}\begin{pmatrix} 1 & 2xq & x^2q^2 \\ 1 & 1+xq & xq \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} Z \\ Z_1 \\ Z_3 \end{pmatrix}(xq)$$

with

$$V=Y-\frac{xyq}{1-xq} \qquad \text{and} \qquad W=Y_1-\frac{xyq}{1-xq}.$$

The generating function $B(x, y, q)$ of polyominoes of $\mathcal{B}$ is given by

$$\begin{pmatrix} B \\ B_1 \\ B_3 \end{pmatrix}(x)=\frac{xyq}{1-xq}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}+\frac{y}{(1-xq)^2}\begin{pmatrix} 1 & 2xq & x^2q^2 \\ 1 & 1+xq & xq \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} B \\ B_1 \\ B_3 \end{pmatrix}(xq). \qquad (6)$$

4. Applications

4.1. Solution of the system

There exists no method to solve systematically these types of equations, called q-difference equations. Nevertheless, iterative methods give interesting developments for the series Y and Z that enumerate respectively directed and convex polyominoes and convex polyominoes. We showed that each term of these developments is the generating function of a particular class of polyominoes. But we have to use other results about stack polyominoes and parallelogram polyominoes, proved by a different method (Bousquet-Mélou 1991, 1992), to completely solve this sytem. We get the following results.

Notation. We use the standard following notation: if $n \geqslant 0$,

$$(a)_n=(1-a)(1-aq)\dots(1-aq^{n-1}).$$

By convention, $(a)_n=1$ if $n \leqslant 0$.

Then, the generating function of directed and convex polyominoes is:

$$Y=y\frac{R(x)-\hat{N}(x)}{N(x)} \qquad (7)$$

where

$$N(x)=\sum_{n\geq 0}\frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_n(yq)_n} \tag{8}$$

$$\hat{N}(x)=\sum_{n\geq 1}\frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_{n-1}(yq)_n} \tag{9}$$

and

$$R(x)=y\sum_{n\geq 2}\left[\frac{x^n q^n}{(yq)_n}\left(\sum_{m=0}^{n-2}\frac{(-1)^m q^{\binom{m+2}{2}}}{(q)_m(yq^{m+1})_{n-m-1}}\right)\right]. \tag{10}$$

The generating function of convex polyminoes is

$$Z=2\sum_{m\geq 1}\frac{y^{m+2}}{(xq)_m^2 N(xq^{m-1})N(xq^m)}(T_{m+1}S(xq^m)-yT_mS(xq^{m+1}))^2$$
$$+\sum_{m\geq 1}\frac{xy^m q^m (T_m)^2}{(xq)_{m-1}(xq)_m} \tag{11}$$

where

$$S(x)=\sum_{n\geq 1}\left(\frac{x^n q^n}{(yq)_n}\sum_{j=0}^{n-1}\frac{(-1)^j q^{\binom{j}{2}}}{(q)_j(yq^{j+1})_{n-j}}\right) \tag{12}$$

$N(x)$ is the series defined by (8) and T_n is the polynomial defined by the following recurrence relations:

$$T_0=1 \qquad T_1=1$$
$$T_n=2T_{n-1}+(xq^{n-1}-1)T_{n-2} \qquad \text{if } n\geq 2. \tag{13}$$

Note that we obtain the same expression for the generating function Y as in Bousquet-Mélou (1992), but a different formula for the Z series.

4.2. *Height and width generating functions*

In the particular case $q=1$, the system (5) becomes easy to solve. We thus get the width and height generating series of all studied subsets of convex polyominoes. As noted in the introduction of the previous paper, they are algebraic series. We find:

$$X(x,y,1)=\frac{1-x-y-\sqrt{\Delta}}{2} \tag{14}$$

$$Y(x,y,1)=\frac{xy}{\sqrt{\Delta}} \tag{15}$$

$$Z(x,y,1)=\frac{xy}{\Delta^2}(1-3x-3y+3x^2+3y^2+5xy-x^3-y^3-x^2y-xy^2-xy(x-y)^2)-\frac{4x^2y^2}{\Delta^{3/2}} \tag{16}$$

$$B(x,y,1)=xy\frac{(1-x)(1-x-2y+y^2-xy)}{(1-x-y)\Delta} \tag{17}$$

with

$$\Delta=1-2x-2y-2xy+x^2+y^2. \tag{18}$$

The first result is standard. The next two have already be proved by Lin and Chang (1988). The fourth one is new.

Note that these series become perimeter generating functions when $x = y$.

Using Lagrange inversion formula and numerous developments in partial fractions, we finally expand these generating functions, in order to get explicit formulae for the number of polyominoes of each class having given height and width, or a given perimeter. We thus prove that the number of parallelogram polyominoes having width p and height q is

$$\frac{1}{p+q-1}\binom{p+q-1}{p}\binom{p+q-1}{q}. \tag{19}$$

The number of directed and convex polyominoes having width p and height q is

$$\binom{p+q-2}{p-1}\binom{p+q-2}{q-1}. \tag{20}$$

The number of convex polyominoes having width p and height q is

$$\frac{pq-1}{p+q-2}\binom{2p+2q-4}{2p-2}-2(p+q-2)\binom{p+q-3}{p-1}\binom{p+q-3}{q-1}. \tag{21}$$

This result has already been proved by Gessel (1990), in a different way. The number of polyominoes of $\mathscr{B}$ having width p and height q is

$$\frac{1}{2}\left(\frac{2p+q-3}{p+q-2}\binom{2p+2q-4}{2p-2}+\binom{p+q-3}{p-1}\right). \tag{22}$$

Expanding perimeter generating functions proves that the number of parallelogram polyominoes having perimeter $2n+2$ is

$$\frac{1}{n+1}\binom{2n}{n}. \tag{23}$$

The number of directed and convex polyominoes having perimeter $2n+4$ is

$$\binom{2n}{n}. \tag{24}$$

The number of convex polyominoes having perimeter $2n+8$ is

$$(2n+11)4^n-4(2n+1)\binom{2n}{n}. \tag{25}$$

The number of polyominoes of $\mathscr{B}$ having perimeter $2n+8$ is

$$6.4^n+2^n. \tag{26}$$

4.3. Area-weighted moments of convex polyominoes

A last application of system (5) is the recursive computation of area-weighted moments of convex (or parallelogram, or directed and convex) polyominoes. These moments are (near to a multiplicative constant) the partial derivatives of $Z(x, y, q)$ with respect to q, evaluated at the point $q=1$. They appear as correction terms when using the finite-lattice method to expand the partition function of the q-state Potts model (Enting and Guttmann 1989).

Let

$$\mathcal{Y}=\begin{pmatrix} Y \\ Y_1 \end{pmatrix} \qquad \text{and} \qquad \mathcal{Z}=\begin{pmatrix} Z \\ Z_1 \\ Z_3 \end{pmatrix}.$$

For any formal series (or vector) $S(x, y, q)$, we denote

$$S^{m,n} \quad \text{for} \quad \frac{\partial^{m+n} S}{\partial x^m \partial q^n}(x, y, 1) \qquad S^{(n)} \quad \text{for} \quad \frac{\partial^n S}{\partial q^n}(x, y, 1)$$

and simply S for $S(x, y, 1)$. Differentiating each equation of (5) n times with respect to q leads to the following system, which allows us to evaluate $\mathcal{Z}^{0,n}$ by induction. Let $\mathbb{1}_A$ denote the characteristic function of the set A. I_n be the identiy matrix of size n:

$$\begin{aligned}(1-x-y-2X)X^{0,n} &= xy\mathbb{1}_{n=1}+nxX^{0,n-1}+(y+X)\sum_{i=1}^{n}\binom{n}{i}x^iX^{i,n-i} \\ &\quad +\sum_{i=1}^{n-1}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}x^jX^{0,n-i}X^{j,i-j} \end{aligned} \tag{27}$$

$$\begin{aligned}[(1-x)I_2-(y+X)N]\mathcal{Y}^{0,n} &= xy\begin{pmatrix}1\\1\end{pmatrix}\mathbb{1}_{n=1}+nx\mathcal{Y}^{0,n-1}+(y+X)N\sum_{i=1}^{n}\binom{n}{i}x^i\mathcal{Y}^{i,n-i} \\ &\quad +\sum_{i=0}^{n-1}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}x^j[X^{0,n-i}N+(n-i)(y+X)^{0,n-1-i}N^{(1)}]\mathcal{Y}^{j,i-j}\end{aligned}$$

$$\begin{aligned}[(1-x)^2I_3-yM]\mathcal{Z}^{0,n} &= [xy(1-2x)\mathbb{1}_{n=1}-2x^2y\mathbb{1}_{n=2}]\begin{pmatrix}1\\1\\1\end{pmatrix}+\mathcal{V}^{(n)}+2nx(1-x)\mathcal{Z}^{0,n-1} \\ &\quad -n(n-1)x^2\mathcal{Z}^{0,n-2} \\ &\quad +yM\sum_{i=1}^{n}\binom{n}{i}x^i\mathcal{Z}^{i,n-i}+nyM^{(1)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i\mathcal{Z}^{i,n-1-i} \\ &\quad +\frac{n(n-1)}{2}yM^{(2)}\sum_{i=0}^{n-2}\binom{n-2}{i}x^i\mathcal{Z}^{i,n-2-i}\end{aligned}$$

with

$$\mathcal{V}^{(n)}=2\sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}\left(\frac{1}{y+X}\right)^{(n-i)}\begin{pmatrix} v^{(j)}v^{(i-j)} \\ v^{(j)}w^{(i-j)} \\ w^{(j)}w^{(i-j)}\end{pmatrix}$$

$$\left(\frac{1}{y+X}\right)^{(n)}=-\frac{1}{y}\left(x\mathbb{1}_{n=1}+\sum_{i=0}^{n}\binom{n}{i}x^iX^{i,n-i}\right)$$

and finally

$$\begin{pmatrix} v^{(m)} \\ w^{(m)} \end{pmatrix} = (1-x)\mathscr{Y}^{0,m} - nx\mathscr{Y}^{0,m-1} - xy\begin{pmatrix} 1 \\ 1 \end{pmatrix}\mathbb{1}_{m\geqslant 1} \qquad \text{if } m \geqslant 0.$$

We thus obtain

$$\frac{\partial Z}{\partial q}(x, y, 1) = \frac{xyP(x, y)}{\Delta^4} + \frac{4x^2y^2(1+x-y)(1-x+y)}{\Delta^{5/2}}$$

with

$$\begin{aligned} P(x, y) = {} & 1-6x-6y+15x^2+15y^2+20xy-20x^3-20y^3-18x^2y-18xy^2+15x^4+15y^4 \\ & -8x^3y-8xy^3+28x^2y^2-6x^5-6y^5+22x^4y+22xy^4-40x^3y^2-40x^2y^3+x^6 \\ & +y^6-12x^5y-12xy^5-5x^4y^2-5x^2y^4+64x^3y^3 \\ & +2xy(x-y)^2(x+y)(x^2+10xy+y^2)+2x^2y^2(x-y)^4. \end{aligned}$$

The calculus of $\mathscr{X}^{0,2}$ is too involved for the computer we use, but, in the special case $x = y = t$, we find

$$\begin{aligned} \frac{\partial^2 Z}{\partial q^2}(t, t, 1) = {} & 2\,\frac{t^3(2+5t-224t^2+1306t^3-3352t^4+4536t^5-3424t^6+1664t^7-512t^8)}{(1-4t)^6} \\ & -2\,\frac{t^4(29-172t+356t^2-312t^3+120t^4)}{(1-4t)^{9/2}}. \end{aligned}$$

These results prove two conjectures of Enting and Guttmann (1989), related to the first two area-weighted moments of convex polyominoes, and subsequently obtained by Lin (1990).

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